Analysis IV

(for EL, GM, MX)

Week 1 17.02.25

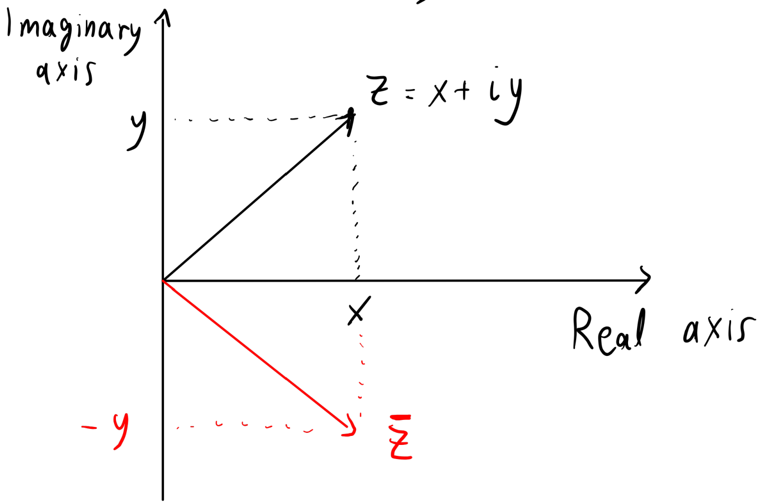
Overview of the course:

- Complex analysis: Complex differentiation, complex integration, Laurent series, Residue theorem
- Review of Fourier transform / series
- Laplace transform (well-suited for functions defined on the half-line $[0,+\infty)$)
- Applications to ODEs / PDEs
  (Initial / boundary value problems).

1. Review of complex numbers $\mathbb{C}$.

- Imaginary unit: i Def: $i^2 = -1$
- $z = x + yi \in \mathbb{C}$, $x, y \in \mathbb{R}$.
  $x = \operatorname{Re} z$, $y = \operatorname{Im} z$
  Real part Imaginary part

This way: Identify $\mathbb{C}$ with $\mathbb{R}^2$



- $z = x + iy \Rightarrow \bar{z} = x - iy$ Complex conjugate

(Reflection across the real axis)

- $|z| = \sqrt{x^2 + y^2}$ (modulus / absolute value)

Geometrically: $|z|$ is distance of (x,y) from $(0,0)$

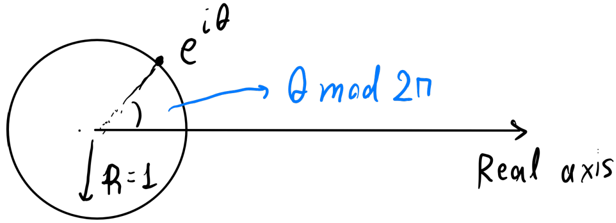
$|z_1 - z_0|$: Distance between z_0, z_1

E.g: $\{z \in \mathbb{C} : |z - 1| < 1\}$ is the open disk centered at $z_0 = 1$ of radius 1.

- Euler's formula:

For $\theta \in \mathbb{R}$, $e^{i\theta} = \cos(\theta) + \sin(\theta)\, i$

So: $|e^{i\theta}| = 1 \Rightarrow e^{i\theta}$ lies on the unit circle



- $\overline{e^{i\theta}} = \overline{\cos\theta + i\sin\theta} = e^{-i\theta}$

- Algebraic operations:

If $z_1 = x_1 + i\, y_1$, $\quad z_2 = x_2 + i y_2$,

- $z_1 + z_2 = (x_1 + x_2) + i\,(y_1 + y_2)$
- $z_1 - z_2 = (x_1 - x_2) + i\,(y_1 - y_2)$

- $z_1 \cdot z_2 = (x_1 + i y_1) \cdot (x_2 + i y_2)$

$$= (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1)\, i$$

- $\boxed{z \cdot \overline{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2}$

$$\bullet \ z_1 / z_2 = \frac{z_1 \cdot \overline{z_2}}{z_2 \cdot \overline{z_2}} = \frac{z_1 \cdot \overline{z_2}}{|z_2|^2} \quad \text{if } z_2 \neq 0$$

Polar expression:

If $z \in \mathbb{C}$, then $z = |z| \cdot e^{i\theta}$ for some $\theta \in \mathbb{R}$.

(So $z^k = |z|^k \cdot e^{ik\theta}$ for $k \in \mathbb{Z}$).

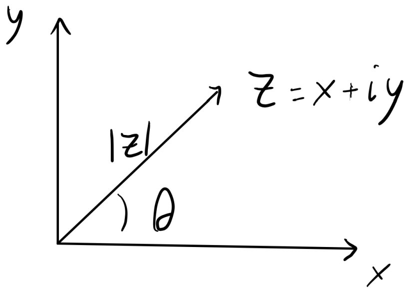
If $z_1 = |z_1| \cdot e^{i\theta_1}$, $z_2 = |z_2| \cdot e^{i\theta_2}$

for some $\theta_1, \theta_2 \in \mathbb{R}$

Then $z_1 = z_2 \iff |z_1| = |z_2|$
and $\theta_1 - \theta_2 \in 2\pi\mathbb{Z}$.

$$z_1 \cdot z_2 = |z_1| \cdot |z_2| \cdot e^{i(\theta_1 + \theta_2)}$$

$$z_1 / z_2 = \frac{|z_1| \cdot e^{i\theta_1}}{|z_2| \cdot e^{i\theta_2}} = \frac{|z_1|}{|z_2|} \cdot e^{i(\theta_1 - \theta_2)}$$



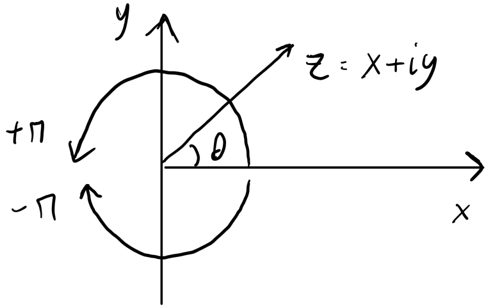
We can always write

$$z = |z| \cdot e^{i\theta}, \quad \theta \in \mathbb{R}$$

- When $z \neq 0$:
  The argument θ is unique up to $2\pi\mathbb{Z}$

- Principal argument: The argument with $-\pi < \theta \leq \pi$

How to find principal argument:



- If $x = 0$: $\theta = +\frac{\pi}{2}$ if $y > 0$

  $\theta = -\frac{\pi}{2}$, $y < 0$

- If $x > 0$:

  $\tan\theta = \frac{y}{x} \Rightarrow \theta = \operatorname{Arctan}\left(\frac{y}{x}\right)$

Ex. If $z = 1 + i$

$$\theta = \operatorname{Arctan}\left(\frac{1}{1}\right) = \frac{\pi}{4} \Rightarrow \underbrace{1 + i = \sqrt{2} \cdot e^{i\frac{\pi}{4}}}_{\text{polar expression}}$$

Remark: Multiplication by $e^{i\theta_0}$

$\Longleftrightarrow$ Rotation by θ_0.

Sometimes, the polar expression is useful in solving algebraic equations.

E.g: Solve

$$z^5 = 2$$

Sol: If $z = |z| \cdot e^{i\theta}$, θ: Principal argument,

then: $z^5 = 2 \Rightarrow |z|^5 \cdot e^{i5\theta} = 2$

$$\Rightarrow \left.\begin{array}{l} |z|^5 = 2 \\ \text{and } \; 5\theta \in 2\pi\mathbb{Z} \end{array}\right\} \begin{array}{l} |z| = 2^{1/5} \\ \theta \in \frac{2\pi}{5}\mathbb{Z} \end{array}$$

But θ: Principal argument: $-\pi < \theta \leq \pi$

$$\Rightarrow \quad \theta \in \left\{ -\frac{4\pi}{5}, -\frac{2\pi}{5}, 0, \frac{2\pi}{5}, \frac{4\pi}{5} \right\}$$

\# 5 solutions

(In general: k-th order equation has k solutions, counted with multiplicity).

2. Complex functions

We will consider functions $f : \mathbb{C} \longrightarrow \mathbb{C}$,

$$z \longrightarrow f(z) = f(x+iy) = \underbrace{u(x,y)}_{\text{Real part}} + \underbrace{v(x,y)}_{\text{Imaginary part}}\, i$$

$$u, v : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

Examples:

- $f(z) = z = x + iy$
- $f(z) = \bar{z} = x - iy$
- $f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + 2xy\, i$
- $f(z) = \dfrac{1}{z} = \dfrac{\bar{z}}{|z|^2} = \dfrac{x - yi}{x^2+y^2} = \dfrac{x}{x^2+y^2} - \dfrac{y}{x^2+y^2}\, i$

  (Defined on $\mathbb{C}/\{0\} = \mathbb{C}^*$)

- $f(z) = e^z = e^{x+iy} = e^x \cdot e^{iy} = e^x \cos y + e^x \sin y \cdot i$

- Trigonometric and hyperbolic trigonometric functions:

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\cosh z = \frac{e^{z} + e^{-z}}{2}, \quad \sinh z = \frac{e^{z} - e^{-iz}}{2}$$

Calculating the real and imaginary parts of $\cos z$:

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = \frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2}$$

$$= \frac{1}{2}\left(e^{-y+ix} + e^{y-ix} \right)$$

$$= \frac{1}{2}\left(e^{-y}(\cos x + i \sin x) + e^{y}(\cos(-x) + i \sin(-x)) \right)$$

Agrees with usual definition when $z \in \mathbb{R}$

$$= \frac{1}{2}(e^{-y} + e^{y}) \cos x + \frac{1}{2}(e^{-y} - e^{y}) \sin x \, i$$

$$= \cosh y \cdot \cos x - \sinh y \cdot \sin x \, i$$

Remark: The rigorous definition of e^z for $z \in \mathbb{C}$ involves assuming that e^z has the same power series expansion as in the case $z \in \mathbb{R}$, i.e.

$$e^z = \sum_{k=0}^{+\infty} \frac{z^k}{k!}$$

3. Limits and Continuity

Def: We will say that $f : \mathbb{C} \longrightarrow \mathbb{C}$ is continuous at a point $z_0 \in \mathbb{C}$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all z with $|z - z_0| < \delta$ we have $|f(z) - f(z_0)| < \varepsilon$.

Equivalently, $\lim_{z \to z_0} f(z) = f(z_0)$.

Made with Goodnotes

Remarks:

1) Let $f(z) = f(x+yi) = u(x,y) + v(x,y)\,i$.

Then: f is continuous at $z_0 = x_0 + y_0 i$

$\iff$ u & v are continuous at (x_0, y_0)

(as functions from $\mathbb{R}^2$ to $\mathbb{R}$)

2) If f, g are continuous, then the same is true for $f \pm g$, $f \cdot g$, $\overline{f}$ etc.

4. Complex derivative

We say that a function $f : \mathbb{C} \longrightarrow \mathbb{C}$ is (complex) differentiable at z_0 if the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$ exists and is finite.

We will denote the limit by $f'(z_0)$.

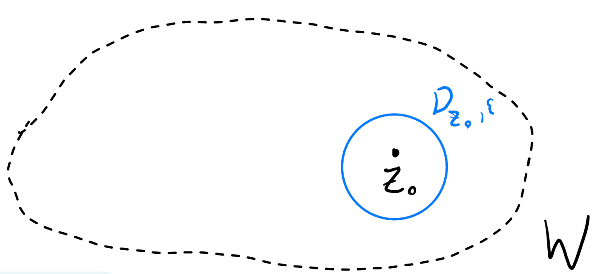
- If f is differentiable at every point of an open set $W \subseteq \mathbb{C}$; f is holomorphic over W
- If f is holomorphic over $\mathbb{C}$: f is entire.

Remark: The same rules for derivatives of products, compositions etc. as in the real case.

Recall: $W \subseteq \mathbb{C}$ is open if $\forall z_0 \in W$, $\exists\, \varepsilon > 0$ such that the disk

$$D_{z_0,\varepsilon} = \{ z \in \mathbb{C} : |z - z_0| < \varepsilon \}$$ is

contained inside W.



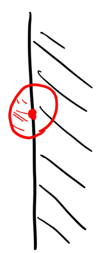
Examples:

- $\mathbb{C}$: Open

- $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$: Open

  Since for $z_0 \in \mathbb{C}^*$: the disk of radius $\varepsilon = \frac{|z_0|}{2}$ does not contain 0, so lies inside $\mathbb{C}^*$

- $\{z \in \mathbb{C} : Re(z) \geq 0\}$ is <u>not</u> open,



  Since for z_0 on the boundary $Re(z_0) = 0$: Any open disc contains points with $Re(z) < 0$, i.e. outside the set.

- If $f(z) = z$: $\forall z_0 \in \mathbb{C}$, $\lim\limits_{z \to z_0} \frac{z - z_0}{z - z_0} = 1$,

  So f is <u>entire</u>.

- If $f(z) = \frac{1}{z}$: $f'(z) = -\frac{1}{z^2}$, so

f <u>holomorphic</u> on $\mathbb{C}^*$.

Next time: Cauchy - Riemann equations.